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1989 J. Phys. A: Math. Gen. 22 4895

(http://iopscience.iop.org/0305-4470/22/22/018)

J. Phys. A: Math. Gen. **22** (1989) 4895-4920. Printed in the UK

Coherent structures in strongly interacting many-body systems: II. Classical solutions and quantum fluctuations

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Received 25 April 1989

Abstract. In the preceding paper we have considered a second-quantised Hamiltonian which can be used to model a large number of important strongly interacting many-body systems. We have shown that the dynamics of such systems can be described exactly using quantum fields. The equations of motion derived in this way are highly non-linear partial differential equations (PDE). In the present paper we use a standard field theoretical approach where initially the fields are treated as classical functions. We focus on the solutions of these equations which may be obtained remarkably and perhaps surprisingly, making full use of very recent mathematical discoveries (the symmetry reduction method). In this work we find exact solutions of the equations of motion for the classical field for each of the four main cases separately found in the preceding paper. These cases may correspond to different physical situations and the last, the second-order case, corresponds to the most general physical situation. Subsequently we will outline possible quantisation procedures for these fields. We then briefly discuss the type of boundary conditions which may be applied to both the classical field and the quantum fluctuations which arise.

1. Introduction

Tuszyński and Dixon (1989, hereafter referred to as I) has been concerned with strongly interacting many-particle systems which can be described using the following second-quantised Hamiltonian:

$$H_{\text{eff}} = \sum_{k,l} \omega_{k,l} q_k^+ q_l + \sum_{k,l,m} \Delta_{k,l,m} q_k^+ q_l^+ q_m q_{k+l-m}. \tag{1}$$

This effective Hamiltonian has been shown to play a prominent role in describing a large number of diverse physical phenomena, especially those associated with symmetry breaking or critical behaviour. It may be used for both bosons and fermions and, in some cases, it can be derived from more complex Hamiltonians which involve interactions between several types of particles.

The method adopted to deal with the model system consisted in first finding equations of motion for second-quantised operators (q_k, q_k^+). These were then re-expressed in terms of a suitably defined quantum field operator ψ. Then interaction constants were expanded in Taylor series of powers of momentum components with respect to a special point in reciprocal space. Since our interest lies in the vicinity of symmetry breaking this special point is so chosen as to correspond to a local energy minimum at a fixed point of the Hamiltonian. Based on this procedure we have distinguished four separate cases.

(*a*) The non-interacting case, where the equation of motion is a linear Schrödinger equation with a dispersive term.

(*b*) The zero-order case where the equation of motion can be written as the cubic non-linear Schrödinger equation.

(*c*) The first-order case where the equation of motion is a generalised non-linear Schrödinger equation with additional non-linear terms involving $\psi^+\psi\nabla\psi$.

(*d*) The second-order case where, in addition to the terms in (*c*), we find terms proportional to $(\nabla_\varepsilon\psi^+)\,\psi(\nabla_\varepsilon\psi)$ and $(\nabla_\varepsilon^2\psi^+)\psi\psi$.

In the general situation anisotropic interactions are manifested by the presence of the signature ε on the ∇_ε and ∇_ε^2 operators, as well as additional residual terms which involve cross derivatives of the independent variables. If all three spatial directions are equivalent either in a fully spherical or cubic geometry these latter terms will disappear and the Laplace–Beltrami operators ∇_ε^2, as well as the gradients ∇_ε, become the standard Laplacian and gradient, respectively.

Following standard methods in quantum field theory (Jackiw 1977, Rajaraman 1987) we proceed to solve the equations derived in I assuming that in the immediate vicinity of the critical point the classical part ϕ of the quantum field ψ dominates (Amit 1978, Ma 1976). Quantum fluctuations may be obtained by linearising ψ about the classical solution ϕ of the equation of motion.

The next four sections will be devoted to seeking classical solutions of the field equations in cases (*a*)–(*d*), respectively. These results have been obtained using the symmetry reduction method and they account for all continuous symmetries in both the Euclidean and Minkowski multidimensional spacetimes (Gagnon and Winternitz 1988, 1989a, b, Winternitz *et al* 1987). Those equations appearing both in (*c*) and (*d*) will be solved by making a special ansatz where $\nabla_\varepsilon\psi$ is represented as a function of ψ. This allows a mapping to be made between these cases and those analysed earlier.

In § 6 the results of quantisation of the classical fields are analysed. This takes into account the various possible geometrical symmetries of the latter as required by the boundary conditions. Those classical fields which are spatially extended may be quantised either through the imposition of periodic boundary conditions or through an $\hbar$ expansion or linearisation. This may bring about band formation in the excitation spectra when the effective potential, after linearisation, is periodic. For localised solutions of the classical field equations quantum fluctuations may be found as bound states in a corresponding linearisation procedure while scattering states, which may also exist in general, highlight the limited stability of the coherent structure.

In the concluding section we discuss some interesting extensions of this work such as the separatrix effect, where quantum fluctuations in a periodic potential may blur the distinction between localised and free states. The constraint of conservation of particle number and the reintroduction of a particular type of statistics will also be discussed.

2. The non-interacting case

As shown in I the equation for the classical field operator ϕ is

$$\mathrm{i}\hbar\partial_t\phi = \nu_0\phi + \mathrm{i}\boldsymbol{\nu}_1\cdot(\nabla_\varepsilon\phi) - \tfrac{1}{2}\nabla_\varepsilon^2\phi \tag{2}$$

where all the parameters have been defined previously and the coordinates in this equation are related to the original coordinates through a rotation and scaling. First of all, the gradient term can be transformed away through a Galilean transformation, namely

$$t' = t \qquad x_i' = x_i - v_i t \qquad i = 1, 2, 3$$

where the propagation velocity is so chosen that $-\hbar v_i = (\nu_1)_i\ \varepsilon_i$. Keeping in mind that the new coordinates are really primed, the equation becomes

$$\mathrm{i}\hbar\partial_t\phi = \nu_0\phi - \tfrac{1}{2}\nabla_r^2\phi. \tag{3}$$

Second, we can separate the time from the space variables by writing

$$\phi = u(t)f(\boldsymbol{r}). \tag{4}$$

Substituting into (3) we find

$$u = u_0 \exp(-\mathrm{i}t\lambda/\hbar) \tag{5}$$

where λ is an arbitrary separation constant and u_0 is also arbitrary. The spatial equation resulting is

$$\nabla^2 f + \Omega f = 0 \tag{6}$$

with $\Omega = 2(\lambda - \nu_0)$ which can be both positive or negative, giving rise to either the Helmholtz or a diffusion-like equation. Both cases are very well known and detailed analyses can be found in many textbooks. There are three important coordinate systems in which (6) is readily solved through separation of variables, namely cartesian, spherical polar and cylindrical coordinates.

It is straightforward to show in cartesian coordinates that, if $f = X(x)Y(y)Z(z)$, solutions of a similar form are obtained for each of X, Y and Z in their respective independent coordinates, each having, in general, a different separation constant μ. For example $X'' = \mu_1 X$ and the full solution ϕ takes the form

$$\phi = u_0 \prod_{i=1}^{3} \{A_i \exp[\mu_i^{1/2}(x_i - v_i t)] + B_i \exp[-\mu_i^{1/2}(x_i - v_i t)]\} \exp(-\mathrm{i}t\lambda/\hbar) \tag{7}$$

where A_i and B_i are arbitrary constants which may be fixed by imposing particular boundary conditions and μ_i are the separation constants. When a particular $\mu_i > 0$ the term in braces in (7) either exponentially increases or decreases with the spatial coordinate. However, for a fixed spatial coordinate, depending on the sign of the velocity v_i, again one term will increase and the other decrease with time. If $\mu_i < 0$ all the terms correspond to oscillatory functions in time and space.

In spherical polar coordinates we put $f = R(r)\Theta(\theta)\Phi(\phi)$ and (6) again separates. The angular part of the solution produces spherical harmonics $Y_l^m(\theta, \phi)$ while the radial component is

$$R = r^{-1/2} Z_{l+1/2}(\Omega t^{1/2}) \tag{8}$$

where Z is a Bessel function (Magnus *et al* 1966). If $\Omega > 0$ the above Bessel function decays with increasing r but if $\Omega < 0$ a Bessel function with an imaginary argument results, so depending on the type, I_ν or K_ν, it will either diverge or decay for large r, respectively. However, when r is small and $\Omega < 0$, K_ν will diverge and I_ν tend to zero as r^ν where the minimum value of ν in (8) is $\nu = \frac{1}{2}$.

One should note that the independent variables r, θ, ϕ are actually functions of time because of our initial transformation, i.e.

$$r(t)=\left(\sum_i (x_i-v_it)^2\right)^{1/2} \qquad \theta(t)=\cos^{-1}\left(\frac{x_3-v_3t}{r(t)}\right)$$

$$\phi(t)=\tan^{-1}\left(\frac{x_2-v_2t}{x_1-v_1t}\right).$$

Hence, for large times and fixed coordinates, both angles will approach constant values θ_0 and ϕ_0 and r becomes very large. Since the indicated time dependence originates from the presence of the gradient term in the original equation we can interpret it as a directional source which makes the solution tend asymptotically to a unidirectional form parallel to $\boldsymbol{v}_1$.

The procedure for cylindrical polar coordinates is analogous and we put $f=P(\rho)\Phi(\phi)U(z)$, and find that $\Phi=\exp(\mp im\phi)$, $U=\exp(\mp lz)$, and

$$P=Z_m[(l^2+\Omega)^{1/2}\rho)]. \tag{9}$$

Once again, the independent variables ρ, ϕ and z are time dependent and we can conclude, in a similar way to the previous case, that the gradient term drives the system into a fixed direction.

The analysis presented in this section has been very straightforward and no surprising results have been obtained. In general, we can see three types of behaviour, namely: oscillatory, damped oscillations and exponential decay which may be realised in three different symmetries. None of these solutions, however, exhibit spatial localisation (free of dispersion) which characterises coherent structures. The latter are known to be associated with critical behaviour in many systems. This motivates us to consider the next level of complexity where non-linear terms begin to appear and play a major role.

3. The zeroth-order case

The non-linear Schrödinger (NLS) equation in $(1+1)$-dimensional spacetime is a well known example of an integrable system and, as such, admits soliton-like solutions, both single and multiple. It has been extensively studied in the past (see, for example, Scott *et al* 1973, Bullough and Caudrey 1980, Lamb 1980, Ablowitz *et al* 1981). Without loss of generality we can use the form

$$i\psi_t+\psi_{xx}+q|\psi|^2\psi=0 \tag{10}$$

since all the parameters in (12) below can be absorbed by scaling independent variables and the linear term can be eliminated by a gauge transformation $\psi\to\psi\exp(i\Lambda t)$. For $q>0$ N-envelope soliton solutions exist (Hirota 1976) with a single soliton in the form of a bell-type sech function. In the $q<0$ case multisoliton solutions have a kink shape with a single soliton envelope proportional to the tanh function. Boyer *et al* (1976) performed a symmetry reduction analysis of the $(1+1)$-dimensional Schrödinger equation with an arbitrary non-linearity. A particularly interesting physical case is that with the envelope depending on $\xi_1=x-v_1t$ and the carrier wave depending on $\xi_2=x-v_2t$ (see (13) below). This situation has been studied by Tuszyński *et al* (1987). A curious property has been discovered that the envelope velocities for solitary-wave-like solutions form two bands separated by a gap of forbidden velocities, reminiscent of electronic bands in periodic solids.

The simplest way of introducing interactions between the quasiparticles is to assume that the interaction term is a constant, keeping all terms up to second order in the one-body coupling constant. This gives the following equation:

$$i\hbar\partial_t\psi = \nu_0\psi + i\boldsymbol{\nu}_1 \cdot (\nabla_\varepsilon\psi) - \tfrac{1}{2}\nabla_\varepsilon^2\psi + \Omega f(\boldsymbol{\eta}_0, \boldsymbol{k}_0, \boldsymbol{m}_0)\psi^+\psi\psi. \tag{11}$$

Using a Galilean transformation similar to that in § 2 the equation can be brought to the form of the standard non-linear (cubic) Schrödinger equation:

$$i\hbar\partial_t\psi = \nu_0\psi - \tfrac{1}{2}\nabla_\varepsilon^2\psi + F_0\psi^+\psi\psi \tag{12}$$

where $F_0 = \Omega f(\boldsymbol{\eta}_0, \boldsymbol{k}_0, \boldsymbol{m}_0)$.

The objective of this section is to extensively analyse the symmetries and possible explicit solutions for both the signatures (ε) appearing in (12).

3.1. *Euclidean signatures (+, +, +)*

Although the NLS in more than (1 + 1) dimensions is not integrable, there exist analytical techniques which allow one to find special types of exact solution so we can treat these cases as 'partially' integrable, i.e. integrable when extra symmetry or boundary conditions are imposed. First of all, in Euclidean space ($\varepsilon \equiv +, +, +$) this equation has recently received considerable attention and the results of symmetry reduction have been published in a series of papers by Gagnon and Winternitz (1988, 1989a, b) in great detail. The method of symmetry reduction for PDE has been described fully in a number of books (Olver 1986, Bluman and Cole 1974, Ovsiannikov 1982, Ibragimov 1985), as well as in a large number of papers in the literature. The method is completely algorithmic and it can be straightforwardly applied in any given case. The main four steps are essentially the following.

(i) Finding the symmetry group of the PDE in question and the Lie algebra of the infinitesimal generators of these symmetry transformations.

(ii) Classifying all the subalgebras and all subgroups having generic orbits of codimension one in the space of independent variables. Similar procedures exist for finding the subalgebras and subgroups with generic orbits of higher codimension.

(iii) The invariants of each subgroup are subsequently found so that by solving a system of first-order PDE one can find the symmetry variables ξ and the prefactors for each of the subgroups. Consequently the PDE is reduced to an ODE for the envelope in the case of orbits with codimension one. In higher codimensions this leads to PDE with a smaller number of independent variables.

(iv) Finally the reduced equations are investigated using various techniques such as the Painlevé technique (Ince 1956) and in many cases explicit solutions may be obtained. In those cases where the ODE fail the Painlevé test one can resort to numerical integration techniques to obtain solutions for specific types of initial conditions. Even when numerical or approximate schemes have to be resorted to, symmetry reduction reveals the inherent symmetries and topologies and therefore provides an important physical insight into the many-body system which is being described. Here we only present the main features of the analysis of Gagnon and Winternitz (1988, 1989a, b). In table 1 the reductions of (12) to an ODE (ordinary differential equation) are summarised. This means that the solutions sought take the form

$$\psi(\boldsymbol{x}, t) = \alpha(\boldsymbol{x}, t)f(\xi) \qquad \text{with} \qquad f(\xi) = M(\xi)\exp(i\chi(\xi)). \tag{13}$$

Hence, the envelope $f(\xi)$ and the carrier wave $\chi(\xi)$ depend on only one variable, namely $\xi = \xi(\boldsymbol{x}, t)$ which is called the symmetry variable, and the form of the prefactor $\alpha(\boldsymbol{x}, t)$ can be found for each symmetry variable separately.

Table 1. Solutions having generic orbits of codimension one in the space of independent variables (x, y, z, t) having the form $\chi(x, t) = \alpha(x, t) f(\xi), f(\xi) = M(\xi) \exp(i\chi(\xi))$ (following Gagnon and Winternitz 1988).

Solution number	$\alpha(x, t)$	ξ	χ	ODE
1	$\exp[i(a\theta - bt)]$	$(x^2+y^2)^{1/2} = \rho$	$S_0 \int \frac{d\rho}{\rho M^2} + \chi_0$	$\ddot{M} - \frac{S_0^2}{\rho^2 M^2} + \frac{\dot{M}}{\rho} - \frac{a^2}{\rho^2} M$ $= (a_0 - b)M + a_1 M^3$ $a \geqslant 0, b \in R$
2	$\exp[\frac{1}{6} i a t(3z - at^2)]$	$z - \frac{1}{2} a t^2$	$S_0 \int \frac{d\xi}{M^2} + \chi_0$	$\ddot{M} - \frac{S_0^2}{M^3} - \frac{1}{2} a \xi M$ $= a_0 M + a_1 M^3$ $a > 0$
3	$\exp(-iat)$	z	$S_0 \int \frac{dz}{M^2} + \chi_0$	$\ddot{M} - \frac{S_0^2}{M^3}$ $= (a_0 - a)M + a_1 M^3$ $a \in R$
4	$\exp(-ibt)$	$(x^2+y^2+z^2)^{1/2} = r$	$S_0 \int \frac{dr}{r^2 M^2} + \chi_0$	$\ddot{M} - \frac{S_0^2}{r^4 M^3} + \frac{2}{r} \dot{M} + bM$ $= a_0 M + a_1 M$ $b \in R$
5	$t^{-1/2} \exp\left[i\left(-a_0 t + \frac{z^2}{4t} + a\theta - \frac{1}{2} b \ln t\right)\right]$	t/ρ^2		Third order $a \geqslant 0, b \geqslant 0$
6	$z^{-1} \exp[-i(a_0 t$ $a\theta + b \ln z)]$	z/ρ	$b > 0$ $b = 0$ $S_0 \int \frac{\xi^2}{M^2(1+\xi^2)^{3/2}} d\xi + \chi_0$	Third order $\xi^2(1+\xi^2)\ddot{M} + \frac{-\xi^6}{(1+\xi^2)^2}$ $\times \frac{S_0^2}{M^3} + \xi(\xi^2 - 2)\dot{M}$ $+ (2 - a^2\xi^2)M = a_1 M^3$ $a \geqslant 0, b \geqslant 0$
7	$t^{-1/2} \exp[i(-a_0 t$ $+ a\theta - \frac{1}{2} b \ln t)]$	t/ρ^2	$\chi = S_0 \int \frac{d\xi}{\xi M^2} + \frac{1}{8\xi} + \chi_0$	$4\xi^3 \ddot{M} + 4\xi^2 \dot{M} - 4\xi \frac{S_0^2}{M^3}$ $+ \left(\frac{1}{2} b - a^2 \xi + \frac{1}{16\xi}\right) M$ $= a_1 M^3$ $a \geqslant 0, b \geqslant 0$
8	$t^{-1/2} \exp\left(i\frac{\rho^2}{4t} - a_0 t - \frac{1}{2} b \ln t\right)\right)$	t/z^2		Third order, $b \geqslant 0$
9	$t^{-1/2} \exp[-i(a_0 t$ $+ \frac{1}{2} b \ln t)]$	t/z^2		Third order, $b \geqslant 0$
10	$\rho^{-1} \exp[-i(a_0 t$ $+ b \ln \rho)]$	$a \ln \rho + \theta$	$b = 0$ $\chi = S_0 \int M^{-2}$	$(a^2+1)\ddot{M} - (a^2+1)M^{-3}$ $\times \exp[4a\xi/(a^2+1)] S_0^2$

Table 1. (continued)

Solution number	$\alpha(x, t)$	ξ	χ	ODE
			$\times \exp[2a\xi/(a^2+1)]\,d\xi$ $+\chi_0$	$-2a\dot{M} + M = a_1 M^3$ $a \geqslant 0,\ b \geqslant 0$
11	$t^{-1/2} \exp\left(i\left(\frac{z^2}{4t} - \frac{b}{2}\ln t - a_0 t\right)\right)$	t/y^2		Third order, $b \geqslant 0$
12	$t^{-1/2} \exp[-i(a_0 t + \frac{1}{2} b \ln t)]$	t/r^2		Third order, $b \geqslant 0$

We now make some comments on the solutions in table 1 which have generic orbits of codimension one. Number 3 describes an axial and time-independent envelope and no damping is present in the prefactor α. Similarly number 2 presents again an axial envelope but accelerating along the z axis ($z \sim t^2$) with no reduction in its amplitude. Numbers 8 and 9 are also axial but they accelerate along the z axis with a time dependence $t^{1/2}$ and their amplitude is damped in proportion to $t^{-1/2}$. Number 11 is analogous except propagation is along the y axis. There are also some cylinderical solutions, namely numbers 5 and 7, where ρ is proportional to $t^{1/2}$, i.e. expanding. As expected, their amplitudes are reduced in proportion to $t^{-1/2}$. On the other hand, the cylindrical solution in number 1 is stationary and has a constant amplitude. There are two spherical solutions, i.e. numbers 4 and 12, one stationary and undamped and one expanding with $t^{1/2}$ and damped as $t^{-1/2}$. Number 6 represents a stationary cone whose amplitude diminishes as z^{-1}. The other (number 10) describes a spiral surface parallel to the z axis and its amplitude is reduced according to ρ^{-1} as it unwinds.

In table 2 we have summarised solutions with generic orbits of codimension two. Number 10 gives just a $(1+1)$-dimensional Schrödinger equation. Hence, the solutions will be non-linear plane waves along the z-axis. A somewhat similar situation can be found in number 4 except the plane surface given by ξ_2, undergoes a rotation as t increases, somewhat like a moving domain wall. Furthermore, number 6 exhibits three-dimensional tilting plane waves. Time-independent reduction can be found in number 9 and the equation has the form of a complex non-linear Klein–Gordon equation in the xy plane. A very similar situation is described by number 5 except one of the independent variables accelerates with time. Cylindrical and spherical time-dependent reductions were obtained in number 8 and 1, respectively. There are several cases with cylindrical symmetry, either stationary as in number 2 or with a time-dependent variable as in numbers 3 and 7. In the latter case we also note the acceleration along the z axis.

The remaining four cases, i.e. numbers 11–14, are rather more difficult to interpret since each of the symmetry variables ξ_1 and ξ_2 describes a different surface. In number 11 we see expanding cylinders and stationary cones and in numbers 12 and 14 we have expanding cylinders and stationary spiral surfaces, while in number 13 cones and spiral surfaces, both of which are stationary. As the reader will easily verify, damping terms appear only in the last four cases.

Obviously, of particular interest are such solutions which can be represented as explicit analytical functions of the independent variables. We have carefully re-grouped

Table 2. Solutions having generic orbits of codimension two in the space of independent variables (x, y, z, t), reducing to a PDE in two variables, having the form $\chi(\boldsymbol{x}, t) = (\xi_1, \xi_2)\,\alpha(\boldsymbol{x}, t)$ (following Gagnon and Winternitz 1988).

Solution number	$\alpha(\boldsymbol{x}, t)$	ξ_1	ξ_2	PDE
1	1	$r = (x^2+y^2+z^2)^{1/2}$	t	$\mathrm{i}\partial_t f + \partial_{rr} f + \frac{2}{r}\partial_r f = a_0 f + a_1 f\lvert f\rvert^2$
2	$\exp\mathrm{i}[(b\theta - at)]$ $\theta = \tan^{-1}(y/x)$	$\rho = (x^2+y^2)^{1/2}$	$z + c\theta = \xi$	$\partial_{\rho\rho} f + \frac{1}{\rho}\partial_\rho f + \left(1+\frac{c^2}{\rho^2}\right)\partial_{\xi\xi} f + \frac{b}{\rho^2}(2\mathrm{i}c\partial_\xi f - bf) = (a_0 - a)f + a_1 f\lvert f\rvert^2$ $a \in R,\ b \geqslant 0,\ c \geqslant 0$
3	$\exp(a\mathrm{i}\theta)$	ρ	$t + b\theta = \xi$	$\mathrm{i}\partial_\xi f + \frac{b^2}{\rho^2}\partial_{\xi\xi} f + \frac{a}{\rho^2} \times (2\mathrm{i}b\partial_\xi f - af) + \partial_{\rho\rho} f + \frac{1}{\rho}\partial_\rho f = a_0 f + a_1 f\lvert f\rvert^2$ $a = 0,\ b \geqslant 0$ or $a > 0,\ b \in R$
4	$\exp[\mathrm{i}(y^2/4t)]$	t	$x - \frac{a}{t}y = \xi$	$\mathrm{i}\partial_t f + \left(1+\frac{a^2}{t^2}\right)\partial_{\xi\xi} f + \frac{\mathrm{i}}{2t} f = a_0 f + a_1 f\lvert f\rvert^2$ $a \geqslant 0$
5	$\exp[\frac{1}{6}\mathrm{i}at(3z - at^2)]$	y	$\xi = z - \frac{1}{2}at^2$	$\partial_{yy} f + \partial_{\xi\xi} f - \frac{1}{2}a\xi f = a_0 f + a_1 f\lvert f\rvert^2$ $a > 0$
6	$\exp\left[\frac{\mathrm{i}}{4t}\left(x^2 + \frac{y^2t^2 - 2bxyt + b^2x^2}{t^2 - b^2 + ct}\right)\right]$	t	$\xi = \frac{a(bx - yt)}{t^2 - b^2 + ct} + z$	$\mathrm{i}\partial_t f + \left[1 + \frac{a^2(b^2+t^2)}{(t^2 - b^2 + ct)}\right]\partial_{\xi\xi} f + \frac{\mathrm{i}}{2t}\left[1 + \frac{(b^2+t^2)}{(t^2 - b^2 + ct)}\right] f = a_0 f + a_1 f\lvert f\rvert^2$ $a > 0,\ b \geqslant 0,\ c \geqslant 0$ or $a = b = 0,\ c \geqslant 0$
7	$\exp\{\mathrm{i}[b\theta + \frac{1}{6}at(3z - at^2)]\}$	ρ	$\xi = z - \frac{1}{2}at^2$	$\partial_{\rho\rho} f + \frac{1}{\rho}\partial_\rho f - \frac{b^2}{\rho^2} f + \partial_{\xi\xi} f - \frac{1}{2}a\xi f = a_0 f + a_1 f\lvert f\rvert^2$ $a > 0,\ b \in R$
8	$\exp[\mathrm{i}(b\theta + z^2/4t)]$	t	ρ	$\mathrm{i}\partial_t f + \frac{1}{\rho}\partial_\rho f + \partial_{\rho\rho} f - \frac{b^2}{\rho^2} f + \frac{\mathrm{i}}{2t} f = a_0 f + a_1 f\lvert f\rvert^2$ $b \geqslant 0$

Table 2. (continued)

Solution number	$\alpha(x, t)$	ξ_1	ξ_2	PDE
9	$\exp(-\mathrm{i}at)$	x	y	$\partial_{xx}f+\partial_{yy}f=(a_0-a)f + a_1 f\lvert f\rvert^2$ $a\in R$
10	1	t	z	$\mathrm{i}\partial_t f+\partial_{zz}f=a_0 f+a_1 f\lvert f\rvert^2$
11	$t^{-1/2}\exp[\mathrm{i}(a\theta-a_0t - \frac{1}{2}b\ln t)]$	t/ρ^2	z/ρ	$\xi_1(\mathrm{i}+4\xi_1)\partial_{\xi_1}f+\xi_1\xi_2\partial_{\xi_2}f + 4\xi_1^3\partial^2_{\xi_1\xi_1}f+\xi_1(1+\xi_2^2) \times\partial^2_{\xi_2\xi_2}f+4\xi_1^2\xi_2\partial_{\xi_1\xi_2}f -(-\frac{1}{2}b+a^2\xi_1+\frac{1}{2}\mathrm{i})f = a_1 f\lvert f\rvert^2$ $a\geqslant 0,\ b\geqslant 0$
12	$t^{-1/2}\exp[\mathrm{i}(z^2/4t - \frac{1}{2}b\ln t-a_0t)]$	t/ρ^2	$a\ln\rho+\theta$	$\xi_1(\mathrm{i}+4\xi_1)\partial_{\xi_1}f+4\xi_1^3\partial_{\xi_1\xi_1}f + \xi_1(1+a^2)\partial_{\xi_2\xi_2}f - 4a\xi_1^2\partial_{\xi_1\xi_2}f + \frac{1}{2}bf=a_1 f\lvert f\rvert^2$ $a\geqslant 0,\ b\geqslant 0$
13	$z^{-1}\exp[-\mathrm{i}(a_0t+b\ln z)]$	z/ρ	$a\ln\rho+\theta$	$\xi_1[\xi_1^2-2(1+b\mathrm{i})]\partial_{\xi_1}f + \xi_1(1+a^2)\partial^2_{\xi_2\xi_2}f -2a\xi_1^3\partial_{\xi_1\xi_2}f + \xi_1^2(1+\xi_1^2)\partial_{\xi_1\xi_1}f + (1+b\mathrm{i})(b\mathrm{i}+2)f=a_1 f\lvert f\rvert^2$ $a\geqslant 0,\ b\geqslant 0$
14	$t^{-1/2}\exp[\mathrm{i}(-\frac{1}{2}b\ln t - a_0t)]$	t/ρ^2	$a\ln\rho+\theta$	$\xi_1(\mathrm{i}+4\xi_1)\partial_{\xi_1}f+4\xi_1^3\partial_{\xi_1\xi_1}f + \xi_1(1+a^2)\partial_{\xi_2\xi_2}f -4a\xi_1^2\partial_{\xi_1\xi_2}f + \frac{1}{2}(b-\mathrm{i})f=a_1 f\lvert f\rvert^2$ $a\geqslant 0,\ b\geqslant 0$

such solutions into five main groups according to their physical form. Within each group the reader will also find a number of specific functional dependences. These are as follows.

(i) Spatially homogeneous solutions (mean field)

$$\psi(\boldsymbol{r}, t)=\psi_0\exp(-\mathrm{i}bt).$$

(ii) Unidirectional quasilinear solutions

$$\psi(\boldsymbol{r}, t)=\psi_0 P_{\mathrm{II}}(c_2\eta+d_2)\exp(\tfrac{1}{6}\mathrm{i}at\eta)$$

where the symmetry variable is $\eta=3z-at^2$ and P_{II} is a Painlevé transcendant (Ince 1956).

(iii) Solutions with cylindrically symmetric envelopes

$$\psi(\boldsymbol{r}, t)=\psi_0\rho^{-1/3}F(\xi)\exp[\mathrm{i}(c_3\theta+d_3t+g_3(\xi))]$$

where the symmetry variable is $\xi = a_3\rho^{2/3} + b_3$, the function F can be any of the following forms and $g_3(\xi)$, in each case, can be found in the original papers:

$$F(\eta) = \text{constant}, \eta^{-1}, \left[\mathrm{cn}\left(\eta, \frac{1}{\sqrt{2}}\right)\right]^{\mp 1}, \mathrm{tn}\left(\eta, \frac{1}{\sqrt{2}}\right), \mathrm{dn}\left(\eta, \frac{1}{\sqrt{2}}\right), \left(\frac{5+\sin\eta}{1-\sin\eta}\right)^{1/2},$$

$$[e_3 + f_3\,\mathrm{sn}^2(\eta, k)]^{1/2}, \frac{[e_3 + f_3\,\mathrm{sn}^2(\eta, k)]^{1/2}}{\mathrm{cn}(\eta, k)},$$

$$\left(\frac{e_3 + f_3\,\mathrm{cn}(\eta, k)}{1 + \mathrm{cn}(\eta, k)}\right)^{1/2}, P_{11}(\eta)$$

where $\mathrm{cn}(\eta, k)$, $\mathrm{sn}(\eta, k)$, $\mathrm{tn}(\eta, 1/\sqrt{2})$ and $\mathrm{dn}(\eta, 1/\sqrt{2})$ are Jacobi elliptic functions with modulus k or its special value $1/\sqrt{2}$, as indicated.

(iv) Stationary z-dependent solutions

$$\psi(\boldsymbol{r}, t) = \psi_0 H(\omega)\exp[\mathrm{i}(c_4 t + g_4(\omega))].$$

In this case the symmetry variable has the form $\omega = a_4 z + b_4$ and $H(\omega)$ can have any of the following forms:

$$\omega^{-1}, \mathrm{cn}^{\mp 1}\left(\omega, \frac{1}{\sqrt{2}}\right), \mathrm{tn}\left(\omega, \frac{1}{\sqrt{2}}\right) \mathrm{dn}\left(\omega, \frac{1}{\sqrt{2}}\right), \left(e_4 + \frac{f_4}{\omega^2}\right)^{1/2},$$

$$\mathrm{cosech}(\omega), \mathrm{sech}(\omega), \sec(\omega), \tan(\omega), \tanh(\omega), \coth(\omega), \mathrm{cn}^{\mp 1}(\omega, k),$$

$$\mathrm{tn}(\omega, k), [e_4 + f_4\,\mathrm{cn}^2(\omega, k)]^{1/2}, \mathrm{sn}(\omega, k),$$

$$\frac{[e_4 + f_4\,\mathrm{sn}^2(\omega, k)]^{1/2}}{\mathrm{cn}(\omega, k)}, [e_4\,\mathrm{cn}^2(\omega, k) + f_4\,\mathrm{sn}^2(\omega, k)]^{1/2},$$

$$\left(\frac{e_4 + f_4\,\mathrm{cn}(\omega, k)}{1 + \mathrm{cn}(\omega, k)}\right)^{1/2}, [e_4 + f_4\sec^2(\omega)]^{1/2}, [e_4 + f_4\,\mathrm{sech}^2(\omega)]^{1/2},$$

$$[e_4 + f_4\,\mathrm{cosech}^2(\omega)]^{1/2}$$

and $g_4(\omega)$ is determined from the form of $H(\omega)$.

(v) Angle-dependent multivalued solutions

$$\psi(\boldsymbol{r}, t) = \psi_0\rho^{-1}B(\Omega)\exp[\mathrm{i}(c_5 t + g_5(\Omega))].$$

Here the symmetry variable is $\Omega = a_5\theta + b_5$ and $B(\Omega)$ may have the following forms:

$$\coth(\Omega), \sec(\Omega), \tanh(\Omega), \mathrm{cn}(\Omega, k), \left(e_5 + \frac{f_5}{\Omega^2}\right)^{1/2}, \mathrm{tn}(\Omega, k),$$

$$\mathrm{sn}(\Omega), \frac{[e_5 + f_5\,\mathrm{sn}^2(\Omega, k)]^{1/2}}{\mathrm{cn}(\Omega, k)}, [e_5 + f_5\,\mathrm{cn}^2(\Omega, k) + h_5\,\mathrm{sn}^2(\Omega, k)]^{1/2},$$

$$\left(e_5 + \frac{f_5 + h_5\,\mathrm{sn}(\Omega, k)}{\mathrm{cn}^2(\Omega, k)}\right)^{1/2}, [e_5 + f_5\sec^2(\Omega)]^{1/2}, [e_5 + f_5\,\mathrm{sech}^2(\Omega)]^{1/2},$$

$$[e_5 + f_5\,\mathrm{cosech}^2(\Omega)]^{1/2}, \left(\frac{e_5 + f_5\,\mathrm{cn}(\Omega, k)}{1 + \mathrm{cn}(\Omega, k)}\right)^{1/2}$$

and $g_5(\omega)$ is found from the particular functional dependence of $B(\Omega)$.

In cases (i)-(v) the parameters a_i, b_i, c_i, d_i, e_i, f_i, h_i, $(i=1,\ldots,5)$, ψ_0, b, a are constants which are *not* in general arbitrary. They are determined from the original PDE and are functions of the constants given by Gagnon and Winternitz (1988, 1989a, b) and Winternitz *et al* (1987) who specified regimes of applicability for each solution's constants. The reader should consult these latter papers for further details. In the above cases, for a particular type of solution, i.e. $1, 2, \ldots, 5$, and each particular function of the symmetry variable, the constants e_i, f_i and h_i will have different values and are listed in the form given for simplicity.

3.2. *Minkowski signatures* $(-, +, +)$

Unfortunately, to the best of our knowledge, there exists no analogous analysis of the (3+1)-dimensional non-linear Schrödinger equation given by (12) for the Minkowski signature. A very simple-minded approach would be to replace those independent variables which correspond to a negative signature, e.g. x_k, by $\mathrm{i}x_k$ and subsequently substitute this transformation into the solutions obtained above. This will not, in general, generate all the solutions since the equation considered is invariant with respect to different symmetry groups for different signatures. It appears though, that only the subgroups of the symmetry group which involve dilation operators (scale-invariant transformations) will be different in the two cases of differing signatures. However, recent studies involving the non-linear Klein-Gordon (NLKG) equation (Winternitz *et al* 1987, 1988, Grundland and Tuszyński 1987) can be used to obtain special solutions in the present case. In order to do this we represent the field as

$$\psi = \eta \exp(\mathrm{i}\chi) \tag{14}$$

where η and χ are real. Substituting (14) into (12) and separating the real and imaginary parts gives for the real part:

$$\nabla_\varepsilon^2 \eta = 2[\nu_0 + \hbar\chi_t + \tfrac{1}{2}(\nabla_\varepsilon \chi)^2]\eta + F_0\eta^3 \tag{15}$$

and for the imaginary part:

$$\hbar\eta_t = +\nabla_\varepsilon\chi \cdot \nabla_\varepsilon\eta + \tfrac{1}{2}(\nabla_\varepsilon^2\chi)\eta = 0. \tag{16}$$

Equation (15) will become a cubic NLKG equation if we require that

$$\hbar\chi_t + \tfrac{1}{2}(\nabla_\varepsilon\chi)^2 = \text{constant}. \tag{17}$$

This latter requirement may easily be satisfied. This may be done, for example, in cartesian coordinates if we put

$$\chi = \boldsymbol{k} \cdot \boldsymbol{x} - \omega t + \chi_0. \tag{18}$$

Similarly in cylindrical coordinates a substitution of the form

$$\chi = k_\rho \rho + k_z z + \omega t + \chi_0 \tag{19}$$

or in spherical coordinates

$$\chi = k_r r - \omega t \tag{20}$$

will both enable (17) to be satisfied. Given that (17) holds, we now have to ensure, with the choice of χ from (18), (19) or (20), that (16) is satisfied. The last term clearly vanishes anyway and the first will also if the envelope is stationary. Thus, we only require the two gradients to be orthogonal and as $\nabla_\varepsilon\psi$ is a constant vector, $\nabla_\varepsilon\eta$ will be a vector on a plane perpendicular to this. Therefore, if we choose η to depend on a set of spatial variables which do not overlap with those which χ depends on, the scalar product of $\nabla_F\chi$ with $\nabla_F\eta$ will then always be zero. This, of course can be done for cartesian as well as cylindrical and spherical coordinate systems. Consequently, (15) becomes a one-, two- or three-dimensional NLKG equation (for the carrier wave χ depending on two, one or no spatial variables, respectively) and a large number of solutions have been found in each of these cases, including localised solutions.

3.2.1. The one-dimensional case. In the one-dimensional case the signature clearly plays no role and all the solutions have been listed in Winternitz *et al* (1988). They involve elliptic waves, both singular and non-singular, and their limiting cases, i.e. trignometric periodic solutions and localised solutions (sech x, tanh x).

The first integral of the one-dimensional version of (15), takes the form

$$D\eta_x^2 = A + B\eta^2 + C\eta^4 \tag{21}$$

where A, B, C and D are constants which, with the exception of A, can easily be related to those in (15). Equation (21) is formally integrated to give

$$x - x_0 = \int_{\eta_0}^{\eta} \frac{\mathrm{d}\eta}{(a + b\eta^2 + c\eta^4)^{1/2}} \tag{22}$$

where x_0 and η_0 are constants pertaining to the initial conditions and a, b and c are given by

$$a = A/D \qquad b = B/D \qquad c = C/D.$$

The solutions in (22) will vary according to the particular choice of values of the constants a, b and c and there are several regions where they assume the same functional form. There are four general cases depending on the signs of b and c. Within each of the cases the value of a determines the functional form of the solution and these can be represented by lines originating at points of intersection of the two curves:

$$P(\eta) = -a \qquad Q(\eta) = b\eta^2 + c\eta^4$$

and entering regions where the solutions are real.

In figure 1 we have illustrated this description for the four cases $c \gtrless 0$ and $b \gtrless 0$ and in the different regimes we have indicated the type of solution one would obtain.

3.2.2. Two dimensions in space. Within this category we have two subcases. The first is when the two spatial coordinates on which the envelope depends have the same signature. Here, the equation to solve is the NLKG equation in $E(2)$ space. The other possibility is, of course, when they have different signatures and we will be solving a NLKG equation in $M(1, 1)$ space. We now draw on the results recently published by Winternitz *et al* (1987, 1988). For the case when

$$2[v_0 + \hbar\chi_t + \tfrac{1}{2}(\nabla_F\chi)^2] = \text{constant} = E_0 \neq 0 \tag{23}$$

in (15) one finds the following reductions. In $E(2)$ the symmetry variables may only be

$$\xi_0 = (|x_i|^2)^{1/2} \qquad \xi_1 = (x_1^2 + x_2^2)^{1/2}$$

with the reduced ODE in the form

$$\eta_{\xi\xi} + (k/\xi)\eta_\xi = \lambda(E_0\eta + F_0\eta^3) \tag{24}$$

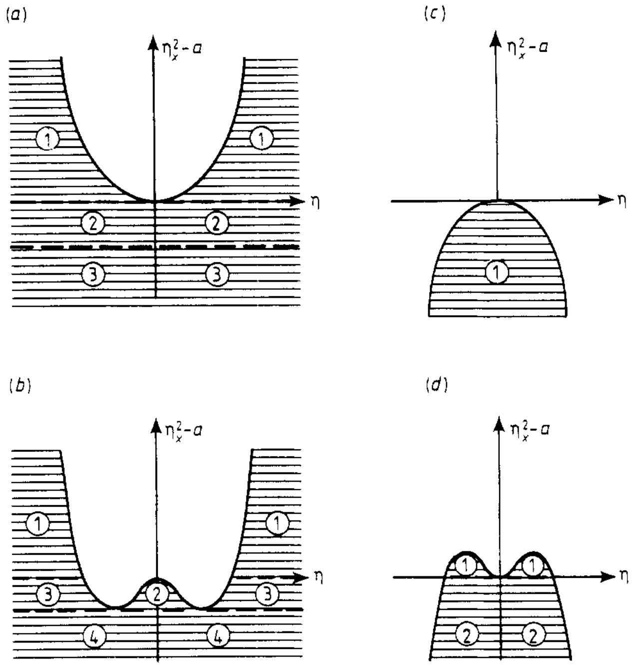

Figure. 1. Graphical illustration of solutions to (21). Solutions take the form: (*a*) when $b>0, c>0$: (1) $\mathrm{cn}^{-1}(x)$; (2) $\mathrm{tn}(x)$; (3) $[(1+\mathrm{cn}(x))/(1-\mathrm{cn}(x))]^{1/2}$; (*b*) when $b<0, c>0$: (1) $\mathrm{cn}^{-1}(x)$; (2) $\mathrm{sn}(x)$; (3) $\mathrm{sn}^{-1}(x)$; (4) $[(1+\mathrm{cn}(x))/1-\mathrm{cn}(x))]^{1/2}$; (*c*) when $b<0, c<0$: (1) $\mathrm{cn}(x)$; (*d*) when $b>0$, $c<0$: (1) $\mathrm{dn}(x)$; (2) $\mathrm{cn}(x)$.

where $\lambda=1$ and $k=1$ or 0 for the two variables ξ_k, respectively. In $M(1,1)$ the symmetry variables allowed are

$$\xi_0=(|x_1|^2)^{1/2} \qquad \xi_1=(x_1^2-x_2^2)^{1/2} \qquad \xi_2=x_1+x_2.$$

In all these cases the reduced ODE is (24) with $k=1$ for ξ_1, and $k=0$ otherwise. The parameter λ is 1 for ξ_1, and -1 otherwise. For $E_0=0$ the symmetry group of the equation acquires scaling transformations in addition to the various rotations and translations present before. This results in a large number of new solutions. However, for cubic non-linearity only the Euclidean case has been exhaustively analysed and further work is in progress to elucidate additional solutions. In the former case the extra reductions are in table 3 following Winternitz *et al* (1988). This table is valid in the case where there are two or three spatial variables and thus for $E(2)$, ξ, in the first row is x_2^2/x_3^2. At present we do not have any results concerning $M(1,1)$ reduction for the subgroups of the similitude group. The reader is referred to the paper by Winternitz *et al* (1987) for a complete analysis of NLKG with quintic non-linearity.

3.2.3. Three dimensions in space. Once again we make use of the previously published results and present them in table 4. In this case $E_0\neq 0$, and the various reductions occur in both Euclidean and Minkowski spaces. In this table all the reductions listed lead to (24) and the values of k and λ for each of the symmetry variables are provided. An important new feature is the existence of so-called degenerate symmetry variables which may depend upon arbitrary functions. The case with $E_0=0$ has been given in

Table 3. The results of symmetry reduction for (15) with $E_0=0$ (see (23)) in two- and three-dimensional Euclidean space where $\eta=\sigma(\boldsymbol{x})f(\xi)$ and $h=$ constant.

$\sigma(\boldsymbol{x})$	ξ	Reduced ODE
$2\left(-\frac{1}{F_0}\right)^{1/2}\frac{1}{x_3}$	$(x_1^2+x_2^2)/x_3^2$	$\xi(\xi+1)\ddot{f}+(1+\frac{5}{2}\xi)\dot{f}+\frac{1}{2}f+f^3=0$
$\left(-\frac{1}{F_0}\right)^{1/2}\frac{1}{x_1}$	x_2/x_1	$(1+\xi^2)\ddot{f}+4\xi\dot{f}+2f+f^3=0$
$2h[-(1/F_0)(h^2+4)(x_1^2+x_2^2)]^{-1/2}$	$\frac{4h}{h^2-4}[-\frac{1}{2}h\ln(x_1^2+x_2^2)^{1/2}$ $+\tan^{-1}(x_2/x_1)]$	$\ddot{f}+\dot{f}+\frac{h^2+4}{4h^2}f+f^3=0$

Table 4. Reduction of (15) with $E_0\neq0$ for $E(3)$ and $M(2,1)$ (following Winternitz *et al* 1987) (p arbitrary).

ξ	k	λ
$E(3)$		
$(x_1^2)^{1/2}$	0	1
$(x_1^2+x_2^2)^{1/2}$	1	1
$(x_1^2+x_2^2+x_3^2)^{1/2}$	2	1
$M(2,1)$		
$(x_1^2)^{1/2}$	0	1
$(x_1^2-x_2^2)^{1/2}$	1	1
$(x_1^2-x_2^2-x_3^2)^{1/2}$	2	1
$(x_3^2)^{1/2}$	0	-1
$(x_2^2+x_3^2)^{1/2}$	1	-1
Degenerate variables $x_2+f(x_1+x_3)$	0	-1
$x_2+p\ln(x_1+x_3)$	0	-1
$x_2+\frac{1}{4}(x_1+x_3)^2$	0	-1

table 3 for a Euclidean metric, and unfortunately the Minkowski metric case has not yet been published when a cubic non-linearity is present.

4. Solutions for the first-order case

Assuming the two-body interaction's coefficient depends linearly on momenta, then, as has been shown in I, this leads to the following equation of motion for the classical field:

$$\mathrm{i}\hbar\partial_t\phi=\nu_0\phi+\mathrm{i}\boldsymbol{\nu}_1\cdot(\nabla_r\phi)-\tfrac{1}{2}\nabla_r^2\phi+\nu_2\phi^+\phi\phi+2\mathrm{i}\boldsymbol{\nu}_3\cdot[\phi^+\phi\nabla_r\phi]. \qquad (25)$$

Following (14) we represent the classical field ϕ in terms of its envelope η and carrier wave χ. When substituted into (25) and real and imaginary parts are equated this produces the following two equations:

$$\tfrac{1}{2}\nabla_F^2\eta = [\hbar\chi_t + \nu_0 - \boldsymbol{\nu}_1\cdot\nabla_F\chi + \tfrac{1}{2}(\nabla_F\chi)^2]\eta + [\nu_2 - 2\boldsymbol{\nu}_3\cdot\nabla_F\chi]\eta^3 \tag{26}$$

$$\hbar\eta_t = (\boldsymbol{\nu}_1 + 2\eta^2\boldsymbol{\nu}_3)\cdot\nabla_F\eta - (\nabla_F\eta)\cdot\nabla_F\chi - \tfrac{1}{2}\eta\nabla_F^2\chi. \tag{27}$$

These equations may be solved by imposing extra constraints, namely attempting to cast (26) into the form of a NLKG equation for which solutions are known (see Winternitz *et al* 1987, 1988). The simplest way of doing this is to require that χ_t and $\nabla_F\chi$ are constants which yields, for χ,

$$\chi = \sum_{i=0}^{3} K_i x_i + \chi_0. \tag{28}$$

Thus the carrier wave is very like a plane wave where, at this stage, the direction of propagation is arbitrary. Hence (26) takes the form of a time-independent three-dimensional cubic NLKG equation:

$$\nabla_F^2\eta = A\eta + B\eta^3 \tag{29}$$

where

$$A = 2\left(\hbar K_0 + \nu_0 - \sum_{i=1}^{3}\nu_{1i}\varepsilon_i K_i + \tfrac{1}{2}\sum_{i=1}^{3}K_i^2\right)$$

and

$$B = 2\left(\nu_2 - 2\sum_{i=1}^{3}\nu_{3i}\varepsilon_i K_i\right).$$

This still has to be consistent with (27). One method of accomplishing this is to demand that

$$\boldsymbol{\nu}_3\cdot\nabla_F\eta = 0 \tag{30}$$

which clearly removes one spatial variable parallel to the direction of $\boldsymbol{\nu}_3$. Equation (27) can now be satisfied in a non-trivial way by choosing η to have the functional dependence

$$\eta = \eta(x_1 - v_1 t, x_2 - v_2 t) \tag{31}$$

where x_1 and x_2 are two linearly independent coordinates in the plane normal to the constant vector $\boldsymbol{\nu}_3$ and v_1 and v_2 are constant velocity components adjusted so that, when (31) is introduced into (27), it is identically satisfied. The result of this procedure is the NLKG equation in two spatial dimensions corresponding to (29). Once again, we refer to the recently published results of Winternitz *et al* (1987, 1988) and a similar analysis which was given in § 3.2.2 of this paper. In addition to solutions like (31) which involve two independent variables and lead to a two-dimensional NLKG, one also finds reductions to an ODE by using the single variables

$$\xi_1 = v_1 x_1 + v_2 x_2 + ct$$

and

$$\xi_2 = v_3\rho - ct$$

where solutions become $\eta(\xi_1)$ and $\eta(\xi_2)$ and represent plane waves in two dimensions or radial cylindrical waves, respectively.

The profiles of the envelope solutions for (29) represent, in general, Jacobi elliptic functions and have periods which depend on the integration constants (the latter may be fixed by imposing appropriate boundary conditions). Particularly interesting special cases are obtained for infinite periods and these are the well studied sech and tanh solitary waves.

5. Second order

Expanding the two-body interaction to second order in the momenta we find that the classical field ϕ satisfies the following equation of motion:

$$\begin{aligned} i\hbar\partial_t\phi &= \mu_0\phi + i\boldsymbol{\mu}_1\cdot(\nabla_r\phi) + \mu_2\nabla_r^2\phi - 2(\nabla_r\phi^+)\cdot\phi(\nabla_r\phi) \\ &\quad + \mu_3\phi^+\phi\phi + 2\boldsymbol{\mu}_4\cdot i\phi^+\phi\nabla\phi + (\nabla_r^2\phi^+)\phi\phi + \phi^+\phi\nabla_r^2\phi. \end{aligned} \tag{32}$$

We again represent ϕ by

$$\phi = \eta\exp(i\chi)$$

and substitute into (32). On equating real and imaginary parts in the resulting equation we find

$$\begin{aligned} &\mu_2(\nabla_r^2\eta) - 2\eta(\nabla_r\eta)^2 + 2\eta^2\nabla_r^2\eta \\ &\quad = \eta(\mu_2(\nabla_r\chi)^2 + \boldsymbol{\mu}_1\cdot\nabla_r\chi - \hbar\chi_t) + \eta^3(2(\nabla_r\chi)^2 - \mu_3 + 2\boldsymbol{\mu}_4\cdot\nabla_r\chi + 2(\nabla_r\chi)^2) \end{aligned} \tag{33}$$

and

$$\hbar\eta_t = (\boldsymbol{\mu}_1 + 2\eta^2\boldsymbol{\mu}_4)\cdot\nabla_r\eta + 2\mu_2(\nabla_r\eta)\cdot(\nabla_r\chi) + \mu_2\eta\nabla_r^2\chi. \tag{34}$$

To solve (33) and (34) we proceed initially in an analogous manner to first order. Equation (34) can be readily satisfied by demanding that χ be a solution of (28) and η is an arbitrary function of symmetry variable(s) such that its gradient is orthogonal to $\boldsymbol{\mu}_4$ and its time dependence balances the right-hand side of (34). Having done this we proceed to solve (33). By virtue of equation (28) the coefficients of η and η^3 become constants and the equation itself closely resembles a two-dimensional NLKG. Hence (33) can be transformed into

$$\nabla_r^2\eta = \frac{[A\eta + B\eta^3 + 2\eta(\nabla_r\eta)^2]}{\mu_2 + 2\eta^2} \tag{35}$$

where A and B are the constant coefficients of η and η^3 appearing on the right-hand side of (33). Equation (35) is clearly of the form

$$\nabla^2\eta = \Phi(\eta, (\nabla\eta)^2). \tag{36}$$

A class of such non-linear PDE of second order has been studied by Grundland *et al* (1982), Cieciura and Grundland (1984) and Grundland (1984). Symmetry reduction analysis for this type of equation has been performed, assuming only that the symmetry variable ξ has the property

$$(\nabla\xi)^2 = f(\xi)$$

i.e. is constant on every level of the function ξ. It is easy to see that this implies that $\nabla^2\xi$ also has this property and, perhaps more interestingly, an arbitrary function of ξ

has the same property as ξ itself, provided the function is invertible. For this class of equation Grundland *et al* (1982) found all the symmetry variables of codimension one in both Euclidean and Minkowski metrics. In our case, of two-dimensional space, the only symmetry variables ξ for the Euclidean and Minkowski cases are listed in table 5. We also give the corresponding values of $(\nabla\xi)^2$ and $\nabla^2\xi$ and the parameters (ε and K) which appear in the reduced ODE which, in all cases, takes the form

$$\varepsilon\eta''+\frac{K\eta'}{\xi}=\frac{A\eta+B\eta^3+2\varepsilon\eta(\eta')^2}{\mu_2+2\eta^2} \tag{37}$$

where the primes denote differentiation with respect to the symmetry variable ξ. We notice in table 5 that two thirds of the cases have $K=0$ so it is of interest to obtain solutions in this case as an example. This may easily be done by writing

$$\eta'=p \qquad \eta''=p\,\mathrm{d}p/\mathrm{d}\eta.$$

Equation (37) then becomes

$$\varepsilon p\,\mathrm{d}p/\mathrm{d}\eta=g(\eta)+f(\eta)p^2 \tag{38}$$

where

$$g(\eta)=\frac{A\eta+B\eta^3}{\mu_2+2\eta^2} \qquad f(\eta)=\frac{2\varepsilon\eta}{\mu_2+2\eta^2}$$

where we suppose $\mu_2+2\eta^2\neq 0$. By changing the dependent variable in (38) to W where $p=W^{-1}$ we find

$$\varepsilon\,\mathrm{d}W/\mathrm{d}\eta=-g(\eta)W^3-f(\eta)W \tag{39}$$

which is a first-order Bernoulli equation whose solution may be written as

$$(W)^{-2}=(\mu_2+2\eta^2)[\tfrac{1}{4}B\ln(\mu_2+2\eta^2)-C]-\tfrac{1}{4}B(2A/B-\mu_2)=(\mathrm{d}\eta/\mathrm{d}\xi)^2=F(\eta)$$

where C is an integration constant. One further integration provides an implicit solution in the form

$$\xi-\xi_0=\int\mathrm{d}\eta\,F(\eta)^{-1/2}. \tag{40}$$

Table 5. Symmetry variables ξ, $(\nabla\xi)^2$, $\nabla^2\xi$, K and ε for (36) for the Euclidean and Minkowski cases in two-dimensional space.

Euclidean $(++)$	ξ	$(\nabla\xi)^2$	$\nabla^2\xi$	K	ε
(a)	ax_1+bx_2, $a^2+b^2=1$	1	0	0	1
(b)	$x_1^2+x_2^2$	1	$1/\xi$	1	1
Minkowski $(+-)$					
(a)	x_1	1	0	0	1
(b)	x_2	-1	0	0	-1
(c)	x_1+x_2	0	0	0	0
(d)	$x_1^2-x_2^2$	1	$1/\xi$	1	1

The integral in (40) can be qualitatively analysed by looking for the roots η_0 of $F(\eta)=0$. If these roots are real they will appear in pairs at $\mp\eta_0$. The form of $F(\eta)$ admits only four cases:

(*a*) no real roots;
(*b*) two symmetrically located distinct single real roots;
(*c*) a pair of double roots;
(*d*) four distinct single real roots.

Denoting $\mu_2+2\eta^2$ by z and $\frac{1}{4}B\ (2A/D-\mu_2)$ by D we find

$$F(\eta)\equiv G(z)=z[\tfrac{1}{4}B\ln z-C]-D=0 \qquad (z>0)$$

or

$$z\ln z=\frac{4}{B}Cz+\frac{4D}{B}.$$

Analysing the form of the solution in (40) in the immediate vicinity of a particular root, it is easy to conclude (by analogy with elliptic integrals) that localised solutions $\eta(\xi)$ can only exist when the corresponding root is a double root whereas periodic solutions occur between two single distinct roots. Case (*a*) above leads to singular solutions only. Cases (*b*) and (*d*) give periodic solutions which correspond to regions of η between two real roots such that $F(\eta)\geqslant 0$. The most interesting case is that of (*c*) and it is indeed possible whenever $4D/B<0$, in which case the double root is located at $z=-4D/B$ and this implies that the integration constant $C=\frac{1}{4}B\ln|4D/B|+1$. Having a pair of double roots we obtain a solution which interpolates between them as ξ varies from $-\infty$ to $+\infty$. This therefore can be interpreted as a kink-type solitary wave. Note also that should $4D/B$ be greater than or equal to 0 we could still formally have a quadruple root at $\eta=0$ but this would mean an infinite integration constant C and the solution itself would become singular at $\xi=\xi_0$. The solution in (40) is clearly an asymptotic solution for the case $K=1$, corresponding to the case $\xi\to\infty$. Moreover, expanding a solution η of (37) for $K=1$ around a constant solution (mean field envelope) leads to a modified Bessel function equation for the small perturbation. Thus, the latter exhibits damped oscillatory behaviour.

For cases with $K=1$ (37) is much less amenable to exact solution. However, there are a number of transformations which may be applied to cast this equation into a form easier to analyse. For example, setting the independent variable to $u=\lambda\ln\xi$ so that

$$\eta=y(\lambda\ln\xi)$$

results in an equation of the form

$$y''=\frac{\exp(2u/\lambda)}{\lambda^2}g(y)+f(y)(y')^2 \tag{41}$$

where primes now indicate differentation with respect to u and

$$g(y)=\frac{Ay+By^3}{\mu^2+2y^2} \qquad f(y)=\frac{2y}{\mu_2+2y^2}. \tag{42}$$

Assuming $y'\neq 0$, (41) may be divided through by y' and directly integrated with respect to u in the limit when $\lambda\to\infty$ when the first term may be neglected. This produces

$$y'=(K_0\mu_2+2y^2)^{1/2} \tag{43}$$

where K_0 is an integration constant. Equation (43) is separable and its integration yields

$$y=\left(\frac{2}{|\mu_2|}\right)^{1/2}\sinh[\sqrt{2}K_0(u-u_0)] \qquad \text{for } \mu_2>0 \tag{44a}$$

$$y=\left(\frac{2}{|\mu_2|}\right)^{1/2}\cosh[\sqrt{2}K_0(u-u_0)] \qquad \text{for } \mu_2<0 \tag{44b}$$

with u_0 an integration constant.

6. Quantisation procedure

It is well known that in critical phenomena classical effects are by far the most dominant sufficiently close to a critical point, so classical solutions are an extremely good approximation. First-order quantum corrections are then of relative magnitude $\hbar$ and can be treated very accurately in a perturbative manner (Amit 1978).

We are aware of the existence of a number of papers where the non-linear Schrödinger equation is considered from a quantum viewpoint using, for example, the inverse scattering technique (Alonso 1984, 1987, Konishi and Wadati 1986, Wadati and Kuniba 1986, Wadati *et al* 1985). However, all these calculations involve one-dimensional equations and therefore would only be applicable to a small fraction of the situations presented in this paper. In order to study quantum mechanical effects we shall adopt a different approach, namely the standard approach in quantum field theory (Jackiw 1977, Rajaraman 1987) where, in the first approximation, quantum effects are ignored and one solves classical field equations in the first instance as we have done earlier in this paper. Subsequently the effects of quantum mechanics are regained by quantising the classical solutions using several possible approaches. These could be carried out using a WKB semiclassical quantisation, or discreteness could be obtained through the imposition of periodic boundary conditions, or by perturbing the quantum fields around their classical counterparts. In general, the classical solutions obtained can be grouped into four categories.

(i) Constant solutions corresponding to the vaccum expectation values of the quantum field and thus, when they become non-zero, signal spontaneous symmetry breaking in the system.

(ii) Static (in their frame of reference), space-dependent solutions, which represent elementary excitations (when they are periodic) or localised condensate envelopes within which quantum states may eventually be bound.

(iii) Time- and space-dependent solutions which correspond to metastable excitations with finite lifetimes or unstable decay patterns.

(iv) Kinks which describe quantum mechanical tunnelling phenomena.

We shall illustrate the quantisation procedure by using the NLS equation in zeroth order, which was obtained in (12), for simplicity. In principle the same approach can be used for both first- and second-order equations as what follows is an elementary linearisation procedure. Starting with (12), for the classical field, we remove the time dependence by substituting

$$\phi=\eta(\boldsymbol{x})\exp(-\mathrm{i}Et/\hbar) \tag{45}$$

to obtain

$$\tfrac{1}{2}\nabla_\varepsilon^2\eta+F_0\eta^3=E_0\eta \tag{46}$$

where $E_0 = E - \nu_0$. We shall seek quantum corrections in the form

$$\psi = \eta_0 + \Lambda \tag{47}$$

where ψ satisfies (46) while η_0 corresponds to $E = 0$. Substituting (47) into (46), and linearising with respect to Λ, gives

$$-\tfrac{1}{2}\nabla_\varepsilon^2\Lambda + 3F_0\eta_0^2\Lambda = E_0\Lambda. \tag{48}$$

In the remainder of this section we illustrate this procedure with several important types of examples, namely elliptic function solutions of (46), hyperbolic, e.g. tanh and sech, and constant solutions. These exemplify three types of physical behaviour which are periodic, localised and mean field, respectively. Although the actual calculations will be in one dimension, extensions to other dimensions are readily constructed. In fact, the one-dimensional results will also become the exact solutions in cylindrical and spherical cases in the asymptotic limit.

6.1. Elliptic solutions of (46)

Following Winternitz *et al* (1988) there exist three general types of non-singular elliptic solutions of (46). They can be obtained by integrating this equation, reducing it to quadratures, and comparing with standard forms listed by Byrd and Friedman (1971).

(i) When $\varepsilon_i = -1$, the polynomial appearing in the first integral of (46) takes the form

$$\eta_0^4 - \frac{2E_0}{F_0}\eta_0^2 + C = 0 \tag{49}$$

and when it has four distinct real roots $\mp\eta_1, \mp\eta_2$ ($|\eta_2|) > |\eta_1|$), then the solution can be written as

$$\eta_0 = \mp\eta_2\,\mathrm{dn}[(-\varepsilon_i F_0)^{1/2}\eta_2(x_i - x_i^0), k]. \tag{50}$$

Here, $k = (1 - \eta_1^2/\eta_2^2)^{1/2}$, x_i is one of the independent variables and C and ξ_0 are integration constants.

(ii) When $\varepsilon_i = -1$ and (49) has two real roots, $\mp\eta_1$, and two purely imaginary roots $\mp i\eta_2$, the solution is

$$\eta_0 = \mp\eta_1\,\mathrm{cn}\{[-\varepsilon_i F_0(\eta_1^2 + \eta_2^2)]^{1/2}(x_i - x_i^0), k\} \tag{51}$$

where

$$k^2 = \eta_1^2/(\eta_1^2 + \eta_2^2).$$

(iii) When $\varepsilon_i = +1$ and (49) has four real roots $\mp\eta_1, \mp\eta_2$ ($|\eta_2| > |\eta_1|$), then the solution is

$$\eta_0 = \mp\eta_1\,\mathrm{sn}[(\varepsilon_i F_0)^{1/2}\eta_2(x_i - x_i^0), k] \tag{52}$$

where $k = \eta_1/\eta_2$

For a graphical illustration of these solutions see figure 1.

These solutions will provide effective potentials for quantum fluctuations as given by (48). However, the subsequent analysis can be done jointly for all three cases since the potential is proportional to η_0^2 and one can use the well known identities:

$$\mathrm{sn}^2(x, k) + \mathrm{cn}^2(x, k) = 1$$
$$k^2\,\mathrm{sn}(x, k) + \mathrm{dn}^2(x, k) = 1. \tag{53}$$

It is easy to show that all of these cases lead to the equation

$$d^2\Lambda/d\alpha^2 = (a + b\,\mathrm{sn}^2(\alpha, k))\Lambda \tag{54}$$

where the constants a and b and the independent variable α are listed in table 6. Putting $b = n(n+1)k^2$, this equation takes the Jacobian form of the generalised Lamé equation. There are two other forms of this equation (see Whittaker and Watson 1963), namely the algebraic form

$$\frac{d^2\Lambda}{d\xi^2} + \left(\frac{\frac{1}{2}}{\xi - e_1} + \frac{\frac{1}{2}}{\xi - e_2} + \frac{\frac{1}{2}}{\xi - e_3}\right)\frac{d\Lambda}{d\xi} = \frac{[n(n+1)\xi + B]\Lambda}{4(\xi - e_1)(\xi - e_2)(\xi - e_3)} \tag{55}$$

and the Weierstrassian form

$$d^2\Lambda/du^2 = [n(n+1)P(u) + B]\Lambda \tag{56}$$

where B is related to a in (54) by

$$B + e_3 n(n+1) = a(e_1 - e_3)$$

and $P(u)$ is the Weierstrass P function.

If n is a positive integer there are $2n+1$ values of B for which (56) has a solution which can belong to any one of four species. When one of these solutions is expanded in descending powers of ξ, the coefficient of the term in $\xi^{n/2}$ being taken as unity, the function obtained is called a Lamé function of degree n of one or other of the four species. The Lamé functions are usually denoted by $E_n^m(\xi)$ where $m = 1, 2, \ldots, 2n+1$. Inserting a series solution of the form

$$\Lambda = \sum_{r=0}^{\infty} b_r(\xi - e_2)^{\frac{1}{2}n - r}$$

into the algebraic form of Lamé's equation gives a recurrence relation among the coefficients b_r each of which is a polynomial in B of degree r. Whittaker and Watson (1963) have provided expressions for the coefficients of B^r and also their sign. If such a series terminates, e.g. if n is even and $b_{\frac{1}{2}n+1} = 0$, then all succeeding coefficients vanish and B is a root of an algebraic equation of degree $\frac{1}{2}n + 1$. There are $\frac{1}{2}n + 1$ real and distinct values of B when n is even and $\frac{1}{2}(n+1)$ when n is odd.

The fact that for each value of n there are $2n+1$ values of m which provide a solution, E_n^m, is very reminiscent of spherical harmonics in a spherical potential and

Table 6. Independent variables and constants from each of (50), (51) and (52) which appear in (54).

Case	α	a	b
(i)	$\eta_2(-\varepsilon_i F_0)^{1/2}(x_i - x_i^0)$	$\dfrac{-2\varepsilon_i(E_0 - 3F_0\eta_2^2)}{\eta_2(-\varepsilon_i F_0)^{1/2}}$	$\dfrac{-6\varepsilon_i F_0 \eta_2 k^2}{(-\varepsilon_i F_0)^{1/2}}$
(ii)	$[-\varepsilon_i F_0(\eta_1^2 + \eta_2^2)]^{1/2}(x_i - x_i^0)$	$\dfrac{-2\varepsilon_i(E_0 - 3F_0\eta_1^2)}{[-\varepsilon_i F_0(\eta_1^2 + \eta_2^2)]^{1/2}}$	$\dfrac{-6\varepsilon_i F_0 \eta_1^2}{[-\varepsilon_i F_0(\eta_1^2 + \eta_2^2)]^{1/2}}$
(iii)	$\eta_2[\varepsilon_i F_0(x_i - x_i^0)]^{1/2}$	$\dfrac{-2\varepsilon_i F_0}{\eta_2(\varepsilon_i F_0)^{1/2}}$	$\dfrac{6\varepsilon_i F_0 \eta_1^2}{\eta_2(-\varepsilon_i F_0)^{1/2}}$

angular momentum quantisation. When equation (54) has periodic solutions they may be designated in the form

$$\begin{Bmatrix} & \mathrm{sn}\,\alpha, & \mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha, & \\ 1, & \mathrm{cn}\,\alpha, & \mathrm{dn}\,\alpha\,\mathrm{sn}\,\alpha, & \mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha\,\mathrm{dn}\,\alpha \\ & \mathrm{dn}\,\alpha, & \mathrm{sn}\,\alpha\,\mathrm{cn}\,\alpha, & \end{Bmatrix} \prod_p (\mathrm{sn}^2\,\alpha - \mathrm{sn}^2\,\alpha_p). \tag{57}$$

Each of the columns inside { } of (57) denotes a species, by analogy with the four species of ellipsoidal harmonics. Each of the eight prefactors in (57) may multiply the product on the right. The α_p are constants.

Complete solutions for an arbitrary value of B (which is to be fixed by appropriate boundary conditions) may be written

$$\Lambda_{\mp} = \prod_{r=1}^{n} \left(\frac{\sigma(a_r \mp u)}{\sigma(u)\sigma(a_r)}\right) \exp\left(\mp u \sum_{r=1}^{n} \zeta(a_r)\right) \tag{58}$$

where σ and ζ are the sigma and zeta functions of Weierstrass where ζ is defined by

$$\mathrm{d}\zeta(z)/\mathrm{d}z = -P(z)$$

and should not be confused with the zeta function of Riemann. In contrast to the solutions of (57) those in (58), obtained for arbitrary values of B, may contain damping factors or indeed may be singular.

6.2. Hyperbolic solutions of (46)

If $-\nu_0/F_0 > 0$, then for special choices of the integration constant C one can obtain localised solutions of (46). These correspond to the limit of elliptic functions when the Jacobi modulus $k \to 1$. There are two different types of these solitary waves, namely:

(i) for $\varepsilon = +1$

$$\eta_0 = \pm\left(\frac{-\nu_0}{F_0}\right)^{1/2} \tanh[(-\nu_0)^{1/2}(x_i - x_i^0)] \tag{59}$$

and

(ii) for $\varepsilon = -1$

$$\eta_0 = \pm\left(\frac{-2\nu_0}{F_0}\right)^{1/2} \mathrm{sech}[(-2\nu_0)^{1/2}(x_i - x_i^0)]. \tag{60}$$

Substituting (59) and (60) into (48) leads to two eigenvalue equations for the two cases (i) and (ii) which both have the form

$$-\mathrm{d}^2\Lambda/\mathrm{d}\alpha^2 + [(L^2 - \omega_n^2) - L(L+1)\,\mathrm{sech}^2\,\alpha]\Lambda = 0 \tag{61}$$

and in both cases $L = 2$ and ν_0 is negative but in case (i),

$$\alpha = (-\nu_0)^{1/2}(x_i - x_i^0) \qquad \omega_n^2 = 2E_n/|\nu_0| \tag{62a}$$

and in case (ii),

$$\alpha = (-2\nu_0)^{1/2}(x_i - x_i^0) \qquad \omega_n^2 = 3 - E_n/|\nu_0|. \tag{62b}$$

This type of Schrödinger equation has been studied extensively, for example, by Morse and Feshbach (1953) and explicit solutions exist involving hypergeometric functions F, i.e.

$$\Lambda = \cosh^{-K}(\alpha) F(K+3, K + \tfrac{1}{2} - \tfrac{5}{2}|K+1|\mathrm{e}^{\alpha}/(\mathrm{e}^{\alpha} + \mathrm{e}^{-\alpha}) \tag{63}$$

where $K = (4 - \omega_n^2)$. The solution in (63) is a well behaved function (finite as $\alpha \to -\infty$) as long as $K - 2$ is a negative integer. This leads to quantised energies, i.e. in case (i)

$$E_n = \frac{(4 - n^2)}{2} |\nu_0|$$

and in case (ii)

$$E_n = (n^2 - 1)|\nu_0|$$

and $n = 1, 2$ in both cases. Solutions corresponding to higher eigenvalues form a continuous spectrum, are unbound and take the form $\exp(in\alpha)$ multiplied by a Jacobi polynomial in $\tanh \alpha$ (Jackiw 1977).

6.3. *Trigonometric solutions of (46)*

The elliptic solutions given in (50)-(52) can also be examined in the other limiting case, namely when $k \to 0$. This is obtained for dnoidal waves when $\eta_2 \to \eta_1$ and gives

$$\mathrm{dn}(u, k) \approx 1 - (k^2 \sin^2 u)/2.$$

Similarly for cnoidal waves this type of solution is obtained when $\eta_1 \to 0$ and

$$\mathrm{cn}(u, k) \simeq \cos u + k^2 \sin u(u - \sin u \cos u)/4$$

whereas for snoidal waves $\eta_1 \to 0$ also and we can use

$$\mathrm{sn}(u, k) \approx \sin u - k^2 \cos u(u - \sin u \cos u)/4.$$

It should be mentioned that the amplitude of all these oscillations will vanish in the actual limit $k = 0$. For small k, but $k \neq 0$, we could estimate the importance of quantum fluctuations with the above approximate formulae or simply look at the small-k limit of the eigenequation (54) which is common to all cases. This gives us a Mathieu equation

$$\mathrm{d}^2\Lambda/\mathrm{d}\alpha^2 + [-(a + \tfrac{1}{2}b) + \tfrac{1}{2}b \cos 2\alpha]\Lambda = 0. \qquad (64)$$

There is a vast literature dealing with this important equation which, because of its effective periodic potential, leads to Bloch-like eigenfunctions and the formation of allowed and forbidden bands. The eigenfunctions are the various Mathieu functions and for a more detailed exposition the reader is referred to the work of Erdélyi (1953).

6.4. *Constant solutions of (46)*

When $k = 0$ these solutions degenerate into constants and hence on substitution into (54) one obtains

$$\mathrm{d}^2\Lambda/\mathrm{d}\alpha^2 = a\Lambda. \qquad (65)$$

The stability properties of the mean phase are determined by the sign of a (i.e. $a > 0$ unstable and $a < 0$ stable). Obviously, when $a > 0$ and large in magnitude a solution possesses an exponential divergence whereas when $a < 0$ one obtains oscillatory solutions.

7. Conclusions

This paper has been devoted to the study of both classical solutions and their quantum corrections for the field equations describing the behaviour of strongly interacting many-body systems at criticality.

In the non-interacting regime we have found classical solutions in the form of oscillatory, damped and exponential functions in planar, cylindrical and spherical geometries. However, the important feature of this order of approximation is that no localisation phenomena were found. In the zeroth-order case a wealth of solutions have been found based largely on the complete symmetry reduction analysis for the Euclidean signature. Spatially homogeneous, quasi-linear, cylindrical and multivalued solutions have been found with a variety of functional forms which were often given by Jacobi elliptic functions. We have also found special types of solution in the Minkowski signature through an ansatz related to the Klein–Gordon equation. An important point in this order of approximation is that infinitesimal symmetry conditions result in a finite number of geometries in which solutions can be obtained. Moreover, this is the first stage at which localisation can be found (as a result of the presence of non-linearity).

In first order we have found special solutions whose carrier waves are of plane wave type and the envelopes are given by either plane waves or cylindrical waves propagating along the direction normal to that of the carrier wave. Generally the envelopes are expressed in terms of elliptic functions.

In second order we have performed a reduction to a two-dimensional equation which can be viewed as a special case of the non-linear Klein–Gordon equation. The analysis of its solutions indicates a clear possibility of localisation.

The last part of this paper has been concerned with quantisation of these classical solutions. Since the method presented studies the physical system in the vicinity of its critical point one expects classical behaviour to be by far the most dominant (Amit 1978, Ma 1976). Quantum effects can be obtained, to a very good degree of approximation, by linearisation about the classical solutions. To this end we have applied a standard field theoretical procedure and investigated the quantisation derived from the classical zeroth-order solutions. Only one-dimensional examples were used to illustrate the procedure. We see no great difficulty in extending this to higher dimensions. In fact, the results obtained in one dimension give the asymptotic dependence of those in higher dimensions. We have found that localised solutions give rise to low lying bounded excitations and a high-energy continuum of states. Periodic solutions, on the other hand, lead to the formation of both allowed and forbidden bands.

Finally, we would like to comment on the possible consequences of the existence of both localised and extended classical solutions. In the finite-volume case the energies of the elliptic wave solutions become infinitesimally close to their localised limiting cases (when $k \to 1$), i.e. to the solitary waves. With a negligible energy change one could therefore cross a boundary between localised and extended states which is called separatrix crossing (Zaslavsky 1985). This may have important implications on the range of existence of chaotic properties in such systems.

The question of apparent arbitrariness in the obtained geometries for multi-dimensional PDE can be resolved very simply by demanding a particular form of the boundary conditions. These can be given in a natural way by the form of the physical system considered, e.g. a cubic crystal lattice, a magnetic spherical microparticle, etc. Once these conditions have been imposed and the reduction to an ODE found, the solutions of this equation may be required to satisfy subsequent periodic boundary equations in the reduced space. This would then bring about discretisation of the allowed frequencies for oscillatory solutions, very much like semiclassical quantisation. Clearly the geometry of the physical system is reflected in both the classical field and the quantum elementary excitations about it.

In particular applications where the physical system has a conserved number N of particles there is an extra constraint to which the obtained solutions should be subjected, namely

$$\int \psi^{+}\psi \, d^{3}x = N.$$

This may lead to the possibility of analysing non-equilibrium phase transitions such as occur in, for example, lasers or even living systems, where the number of particles may be time dependent as a result of external pumping. We hope to investigate this problem in a future publication.

In conclusion it appears that the wealth of solutions obtained is a clear indication of their potential use in a very wide range of systems in condensed matter physics.

Acknowledgments

One of us (JMD) expresses his gratitude to the Science and Engineering Council for a travel grant. He would also like to thank the Faculty members and staff of the Department of Physics and Mrs J Harries for her hospitality and kindness during his stay at the University of Alberta. The other author (JAT) acknowledges financial support for this research from the Natural Sciences and Engineering Research Council of Canada.

Both authors wish to express their appreciation to Mrs Arlene Cowan for her patience, forbearance and accurate typing of this and a number of other manuscripts.

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